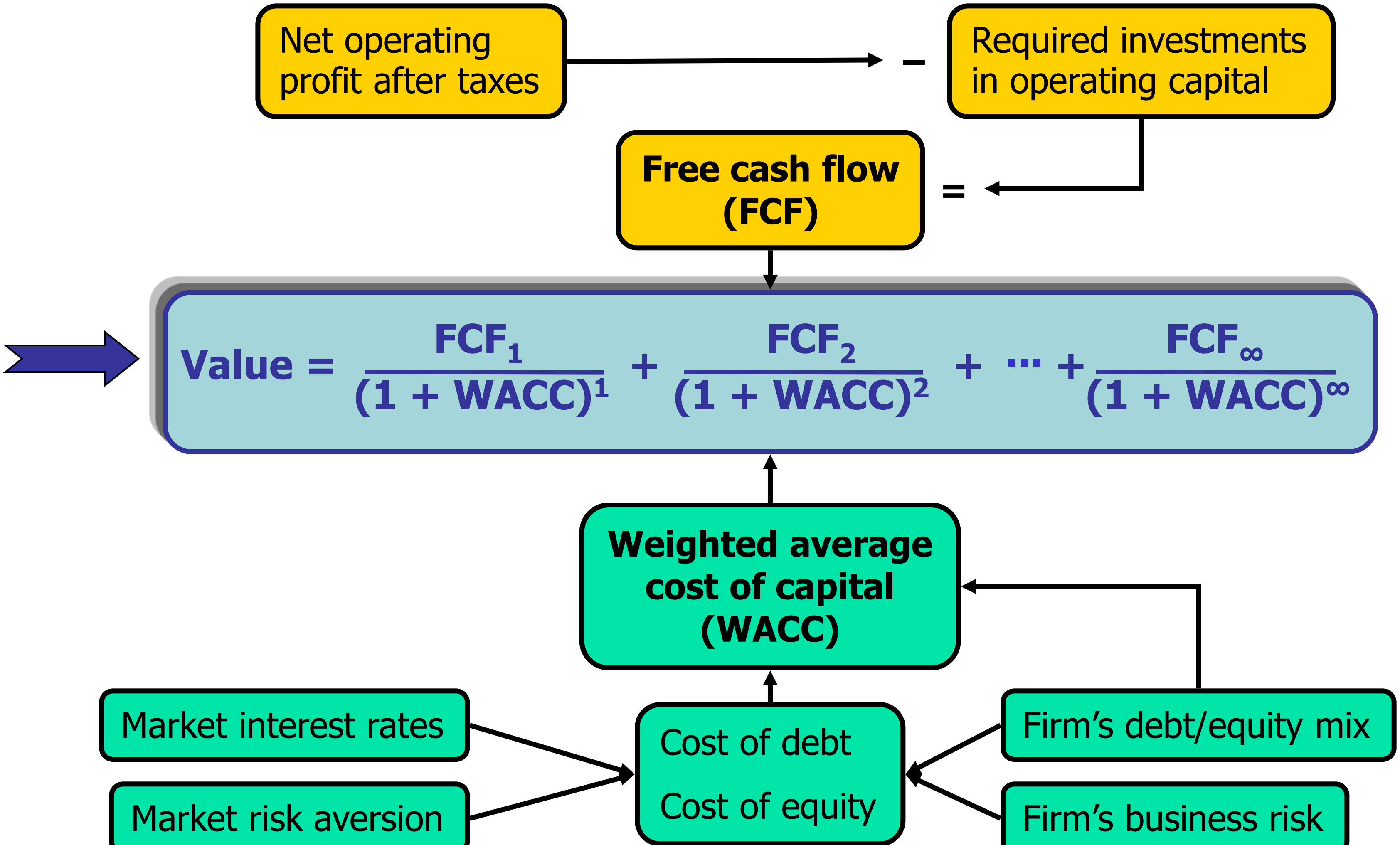
Chapter 4

Time Value of Money

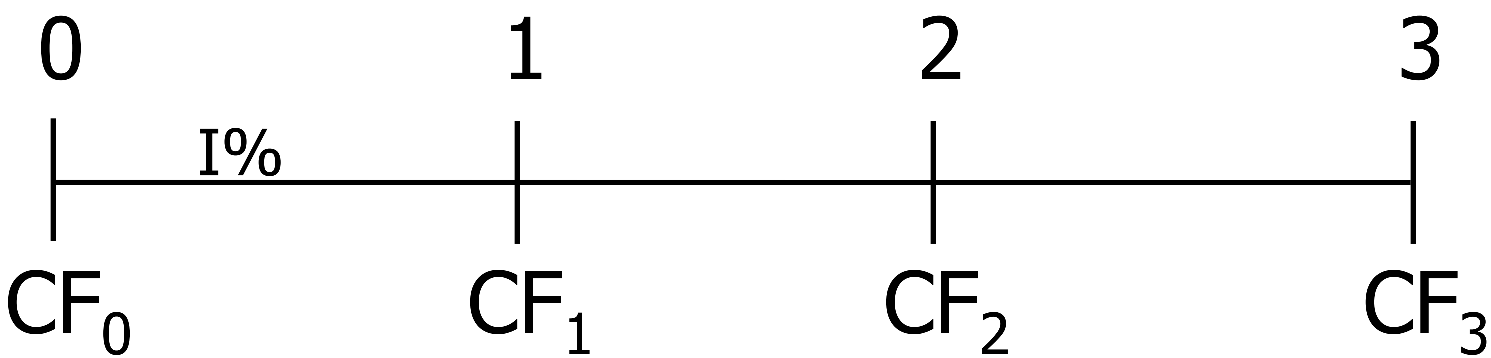
Time Value Topics

- Future value
- Present value
- Rates of return
- Amortization

Determinants of Intrinsic Value: The Present Value Equation

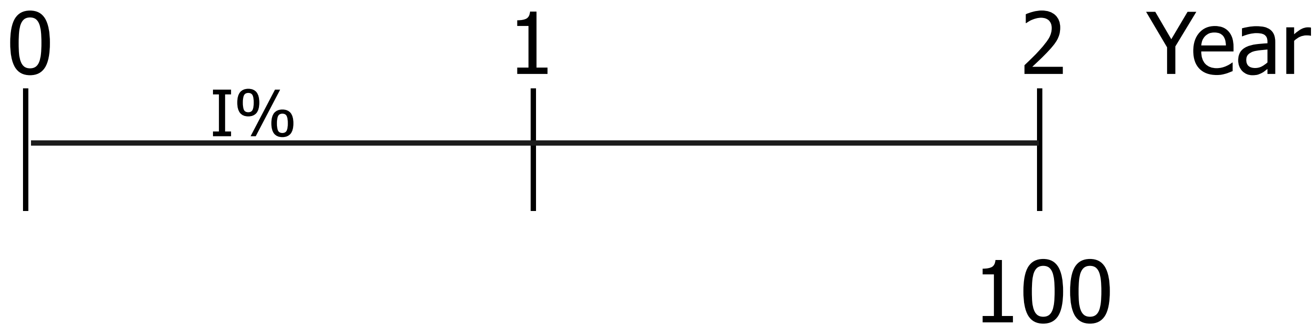


Time lines show timing of cash flows.

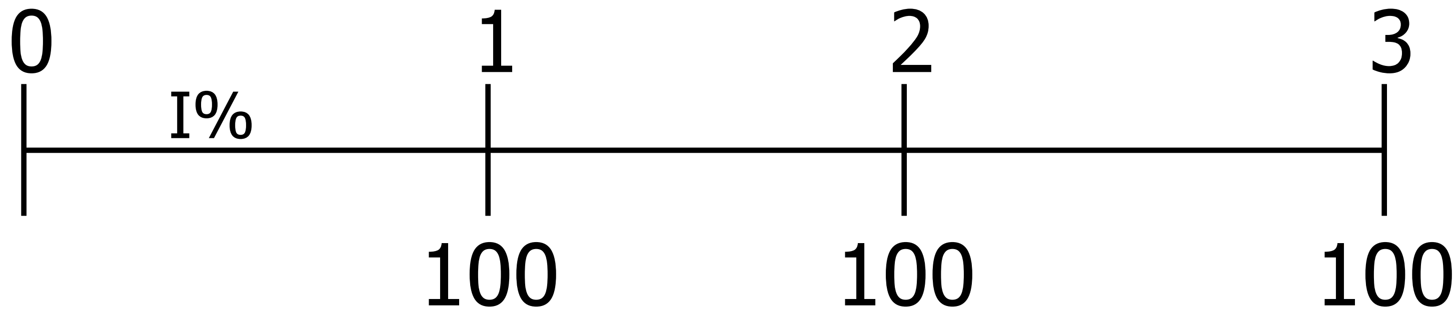


Tick marks at ends of periods, so Time 0 is today; Time 1 is the end of Period 1; or the beginning of Period 2.

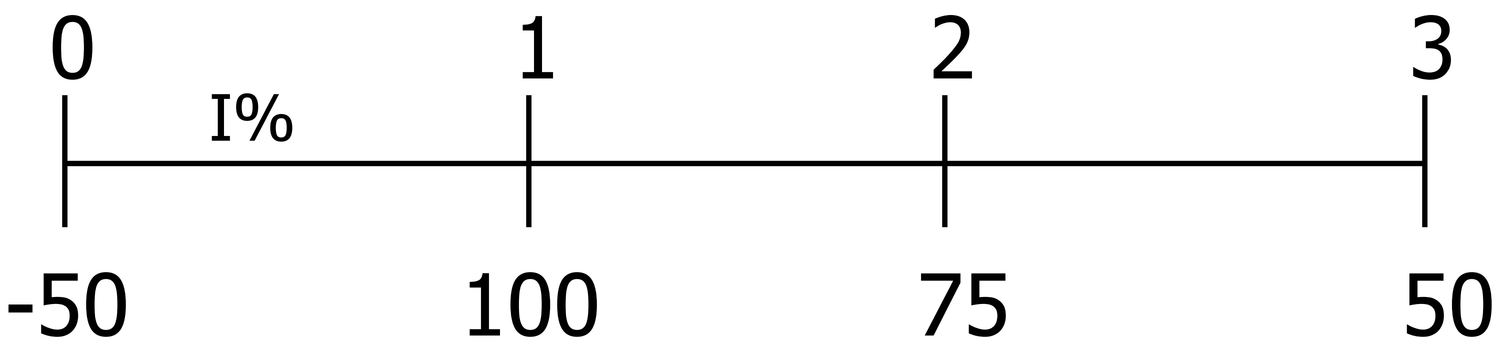
Time line for a $100 lump sum due at the end of Year 2.



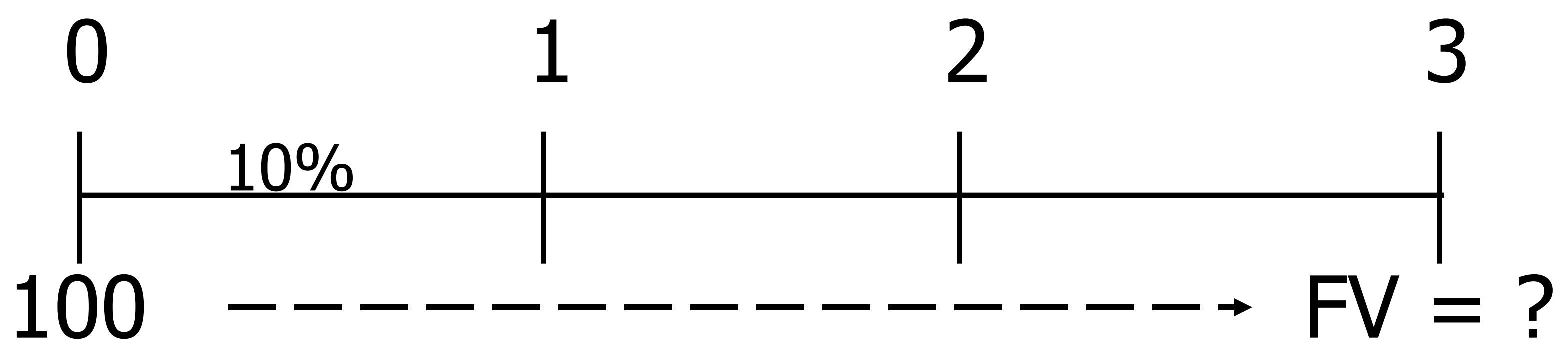
Time line for an ordinary annuity of $100 for 3 years



Time line for uneven CFs



FV of an initial \$100 after 3 years (I = 10%)



Finding FVs (moving to the right on a time line) is called compounding.

After 1 year

$$
\begin{aligned}
FV_1 &= PV + INT_1 = PV + PV\,(I) \\
&= PV(1 + I) \\
&= \$100(1.10) \\
&= \$110.00
\end{aligned}
$$

After 2 years

$$FV_2 = FV_1(1+I)$$
$$= \$110(1.10)$$
$$= \$121.00$$

Or

$$FV_2 = PV(1 + I)(1+I)$$
$$= PV(1+I)^2$$
$$= \$100(1.10)^2$$
$$= \$121.00$$

After 3 years

$$FV_3 = FV_2(1+I)$$
$$= \$121.00(1.10)$$
$$= \$133.10$$

Or

$$FV_3 = FV_2(1+I) = PV(1 + I)^2(1+I)$$
$$= PV(1+I)^3$$
$$= \$100(1.10)^3$$
$$= \$133.10$$

Using the General Formula: $FV_N = PV(1 + I)^N$

Generalizing the approach from previous slides:

$$FV_N = PV(1 + I)^N$$
$$FV_3 = \$100(1.10)^3$$
$$= \$133.10$$

Four Ways to Find FVs

- Step-by-step approach using time line (as shown in previous slides).
- Solve the equation with a regular calculator (formula approach).
- Use a financial calculator.
- Use a spreadsheet.

Financial calculator: HP 10bII+

- Adjust display brightness: hold down ON and push + or –.
- Set number of decimal places to display: Orange Shift key, then DISP key (in orange), then desired decimal places (e.g., 3).
- To temporarily show all digits, hit Orange Shift key, then DISP, then =.

HP 10bII+ (Continued)

- To permanently show all digits, hit ORANGE shift, then DISP, then . (period key).
- Set decimal mode: Hit ORANGE shift, then ./, key. Note: many non-US countries reverse the US use of decimals and commas when writing a number.

HP 10bII+: Set Time Value Parameters

- To set END (for cash flows occurring at the end of the year), hit ORANGE shift key, then BEG/END.
- To set 1 payment per period, hit 1, then ORANGE shift key, then P/YR.

Financial calculator: BAII+

- Set number of decimal places to display: 2nd Format; use the up and down arrows to display DEC=; press 9; press ENTER
- Set AOS calculation; 2nd Format; down arrow 4 times until you see Chn (if you see AOS then just stop and hit CE/C, you are done); 2nd SET (AOS should display); CE/C you are done.

BAII +: Set Time Value Parameters

- To set END (for cash flows occurring at the end of the year), hit 2nd BGN; 2nd SET will toggle between cash flows at the beginning of the year (BGN) and end of the year (END). Leave it as END.
- To set 1 payment per period, hit 2nd P/Y 1 ENTER.

BAII+

- To reset TVM calculations; 2nd CLR TVM.
- To reset cash flow register; CF; 2nd CLR Work.

Financial Calculator Solution

Financial calculators solve this equation:

$$FV_N + PV\,(1+I)^N = 0.$$

There are 4 variables. If 3 are known, the calculator will solve for the 4th.

Here's the setup to find FV

INPUTS	3	10	-100	0	
	N	I/YR	PV	PMT	FV
OUTPUT					133.10

Clearing automatically sets everything to 0, but for safety enter PMT = 0.

Set: P/YR = 1, END.

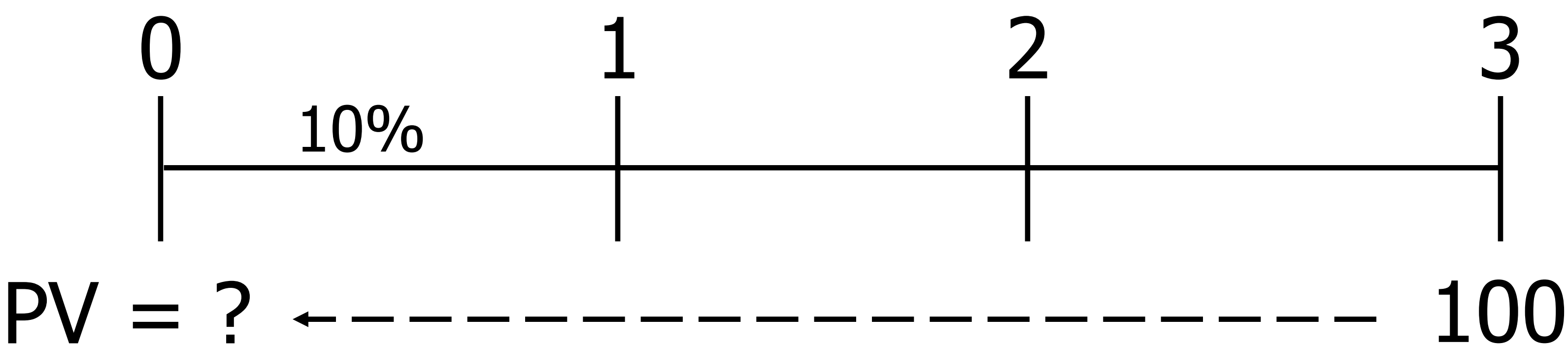
Spreadsheet Solution

- Use the FV function: see spreadsheet in *Ch28 Mini Case.xls*

- = FV(I, N, PMT, PV)
- = FV(0.10, 3, 0, -100) = 133.10

What's the PV of $100 due in 3 years if I/YR = 10%?

Finding PVs is discounting, and it's the reverse of compounding.



Solve $FV_N = PV(1 + I)^N$ for PV

$$PV = \frac{FV_N}{(1+I)^N} = FV_N \left(\frac{1}{1+I}\right)^N$$

$$PV = \$100\left(\frac{1}{1.10}\right)^3$$

$$= \$100(0.7513) = \$75.13$$

Financial Calculator Solution

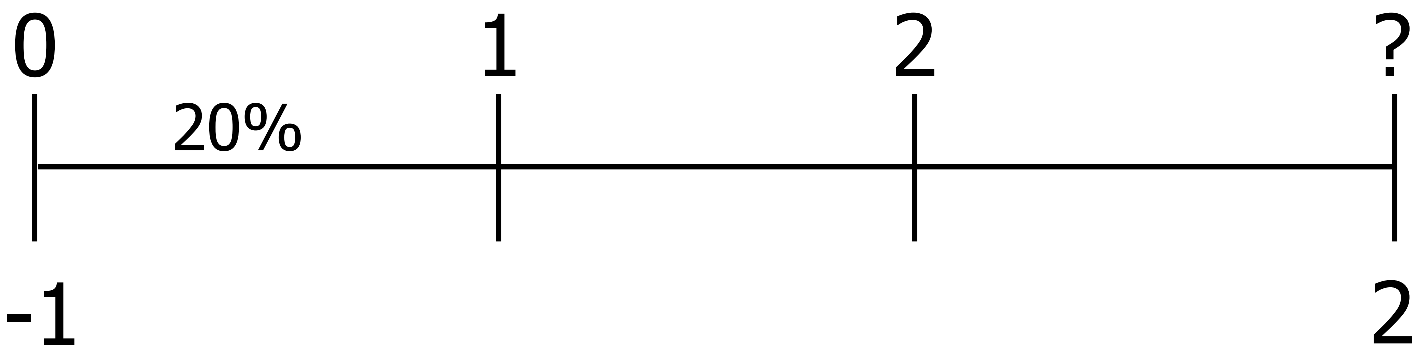
INPUTS	3	10		0	100
	N	I/YR	PV	PMT	FV
OUTPUT			-75.13		

Either PV or FV must be negative. Here PV = -75.13. Put in $75.13 today, take out $100 after 3 years.

Spreadsheet Solution

- Use the PV function: see spreadsheet in *Ch04 Mini Case.xls*
- = PV(I, N, PMT, FV)
- = PV(0.10, 3, 0, 100) = -75.13

Finding the Time to Double



$$FV = PV(1 + I)^N$$

Continued on next slide

Time to Double (Continued)

$$\$2 = \$1(1 + 0.20)^N$$
$$(1.2)^N = \$2/\$1 = 2$$
$$N \, LN(1.2) = LN(2)$$
$$N = LN(2)/LN(1.2)$$
$$N = 0.693/0.182 = 3.8$$

Financial Calculator Solution

	N	I/YR	PV	PMT	FV
INPUTS		20	-1	0	2
OUTPUT	3.8				

Spreadsheet Solution

- Use the NPER function: see spreadsheet in *Ch04 Mini Case.xls*

- = NPER(I, PMT, PV, FV)

- = NPER(0.10, 0, -1, 2) = 3.8

Finding the Interest Rate

$$FV = PV(1 + I)^N$$
$$\$2 = \$1(1 + I)^3$$
$$(2)^{(1/3)} = (1 + I)$$
$$1.2599 = (1 + I)$$
$$I = 0.2599 = 25.99\%$$

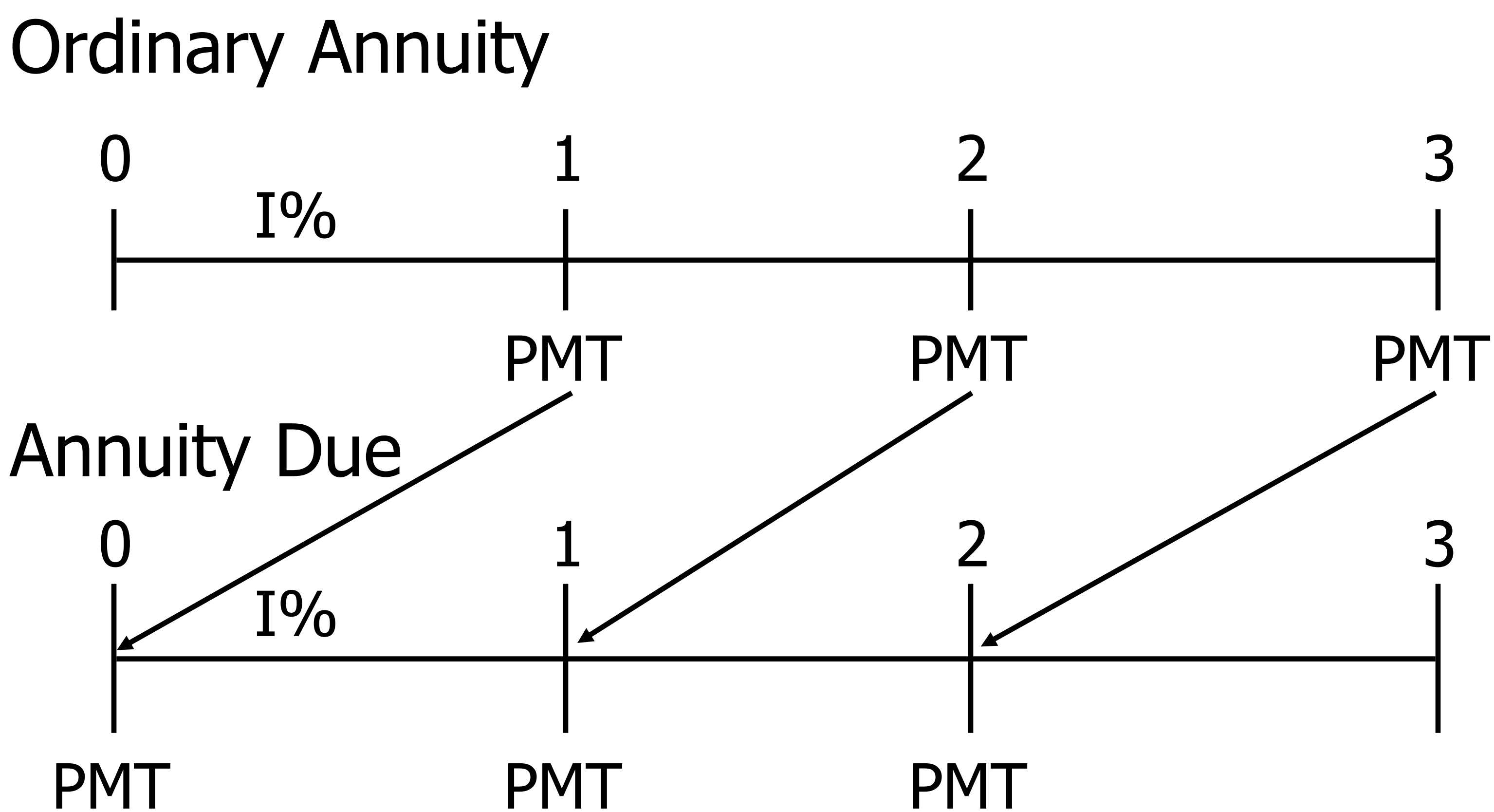
Financial Calculator

	N	I/YR	PV	PMT	FV
INPUTS	3		-1	0	2
OUTPUT		25.99			

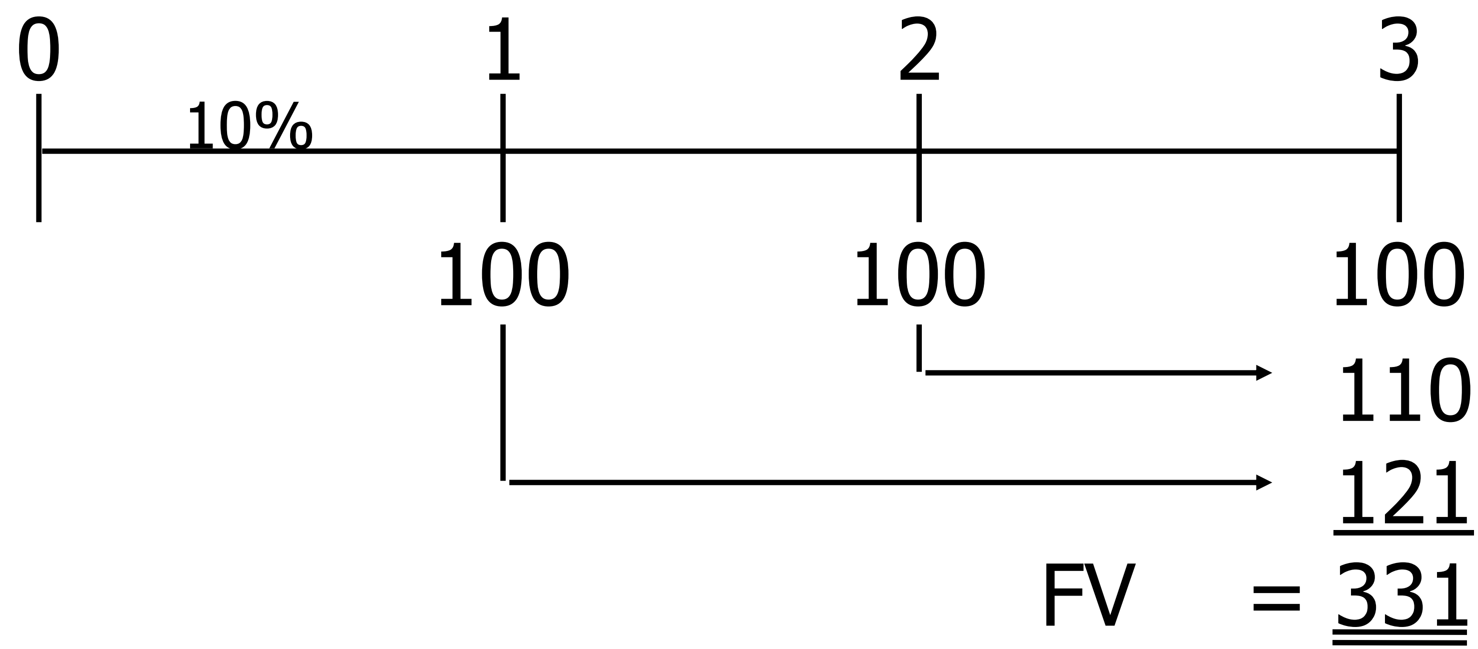
Spreadsheet Solution

- Use the RATE function:
- = RATE(N, PMT, PV, FV)
- = RATE(3, 0, -1, 2) = 0.2599

Ordinary Annuity vs. Annuity Due



What's the FV of a 3-year ordinary annuity of $100 at 10%?



FV Annuity Formula

- The future value of an annuity with N periods and an interest rate of I can be found with the following formula:

$$= PMT\frac{(1+I)^N-1}{I}$$

$$= \$100\frac{(1+0.10)^3-1}{0.10} = \$331$$

Financial Calculator Formula for Annuities

- Financial calculators solve this equation:
- $$\mathbf{PV(1+I)^{N} + PMT\frac{(1+I)^{N}-1}{I} + FV_{N} = 0}$$
- There are 5 variables. If 4 are known, the calculator will solve for the 5th.

Financial Calculator Solution

INPUTS	3	10	0	-100	
	N	I/YR	PV	PMT	FV
OUTPUT					331.00

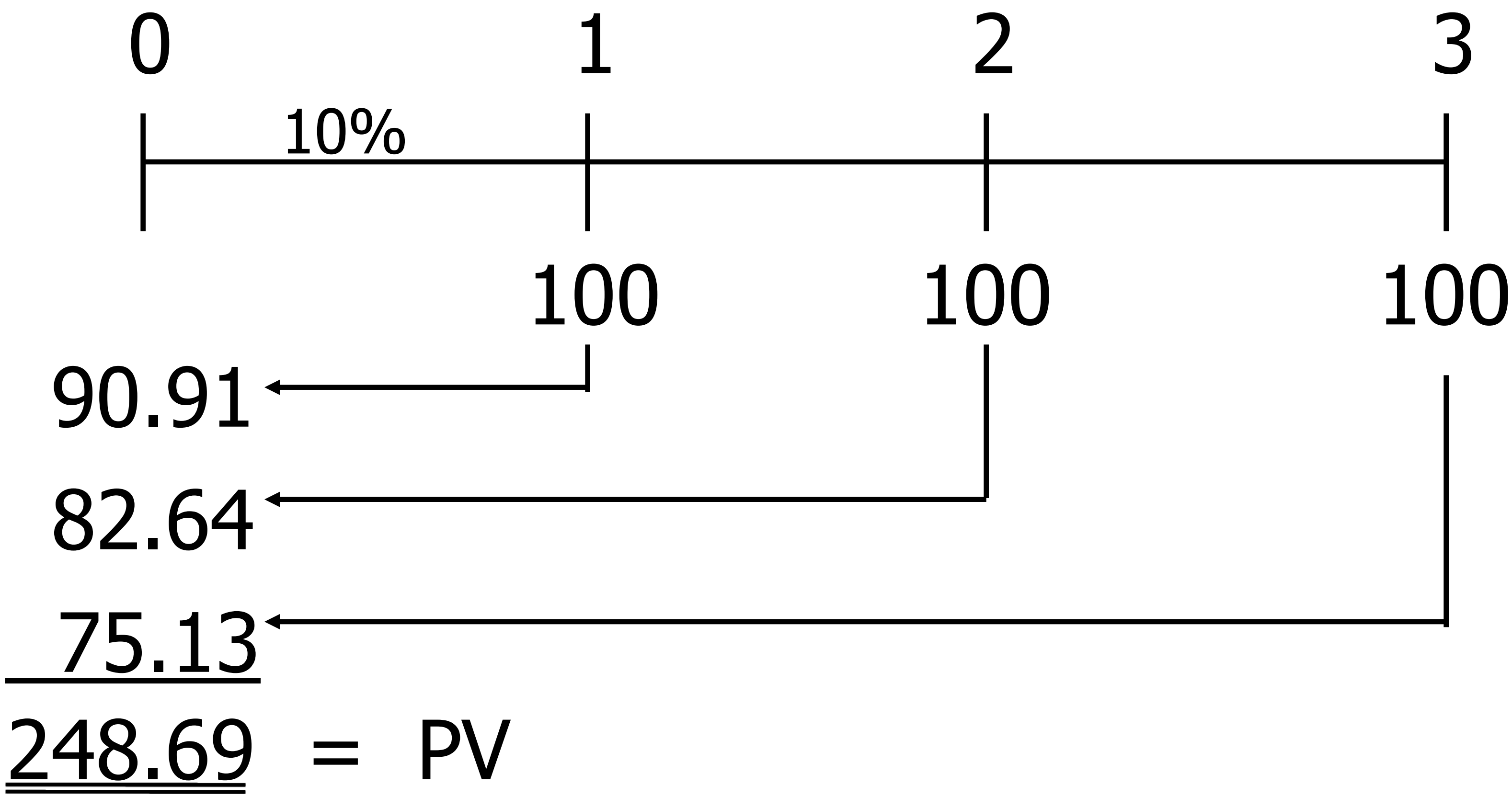
Have payments but no lump sum PV, so enter 0 for present value.

Spreadsheet Solution

- Use the FV function: see spreadsheet.

- = FV(I, N, PMT, PV)
- = FV(0.10, 3, -100, 0) = 331.00

What's the PV of this ordinary annuity?



PV Annuity Formula

- The present value of an annuity with N periods and an interest rate of I can be found with the following formula:

$$= \text{PMT}\left(\frac{1}{\text{I}} - \frac{1}{\text{I}\,(1+\text{I})^{\text{N}}}\right)$$

$$= \$100\left(\frac{1}{0.1} - \frac{1}{0.1(1+0.1)^{3}}\right) = \$248.69$$

Financial Calculator Solution

INPUTS	3	10		100	0
	N	I/YR	PV	PMT	FV
OUTPUT			-248.69		

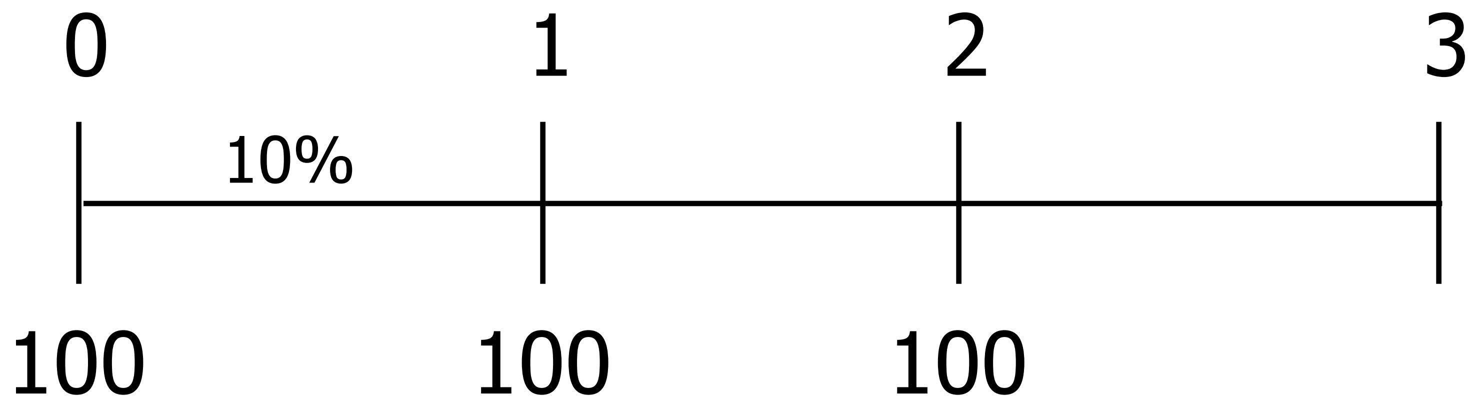
Have payments but no lump sum FV, so enter 0 for future value.

Spreadsheet Solution

- Use the PV function: see spreadsheet.

- = PV(I, N, PMT, FV)
- = PV(0.10, 3, 100, 0) = -248.69

Find the FV and PV if the annuity were an annuity due.



PV and FV of Annuity Due vs. Ordinary Annuity

- PV of annuity due:
- = (PV of ordinary annuity) (1+I)
- = ($248.69) (1+ 0.10) = $273.56

- FV of annuity due:
- = (FV of ordinary annuity) (1+I)
- = ($331.00) (1+ 0.10) = $364.10

PV of Annuity Due: Switch from “End” to “Begin”

BEGIN Mode

INPUTS	3	10		100	0
	N	I/YR	PV	PMT	FV
OUTPUT			-273.55		

FV of Annuity Due: Switch from "End" to "Begin"

BEGIN Mode

INPUTS	3	10	0	100	
	N	I/YR	PV	PMT	FV
OUTPUT					-364.10

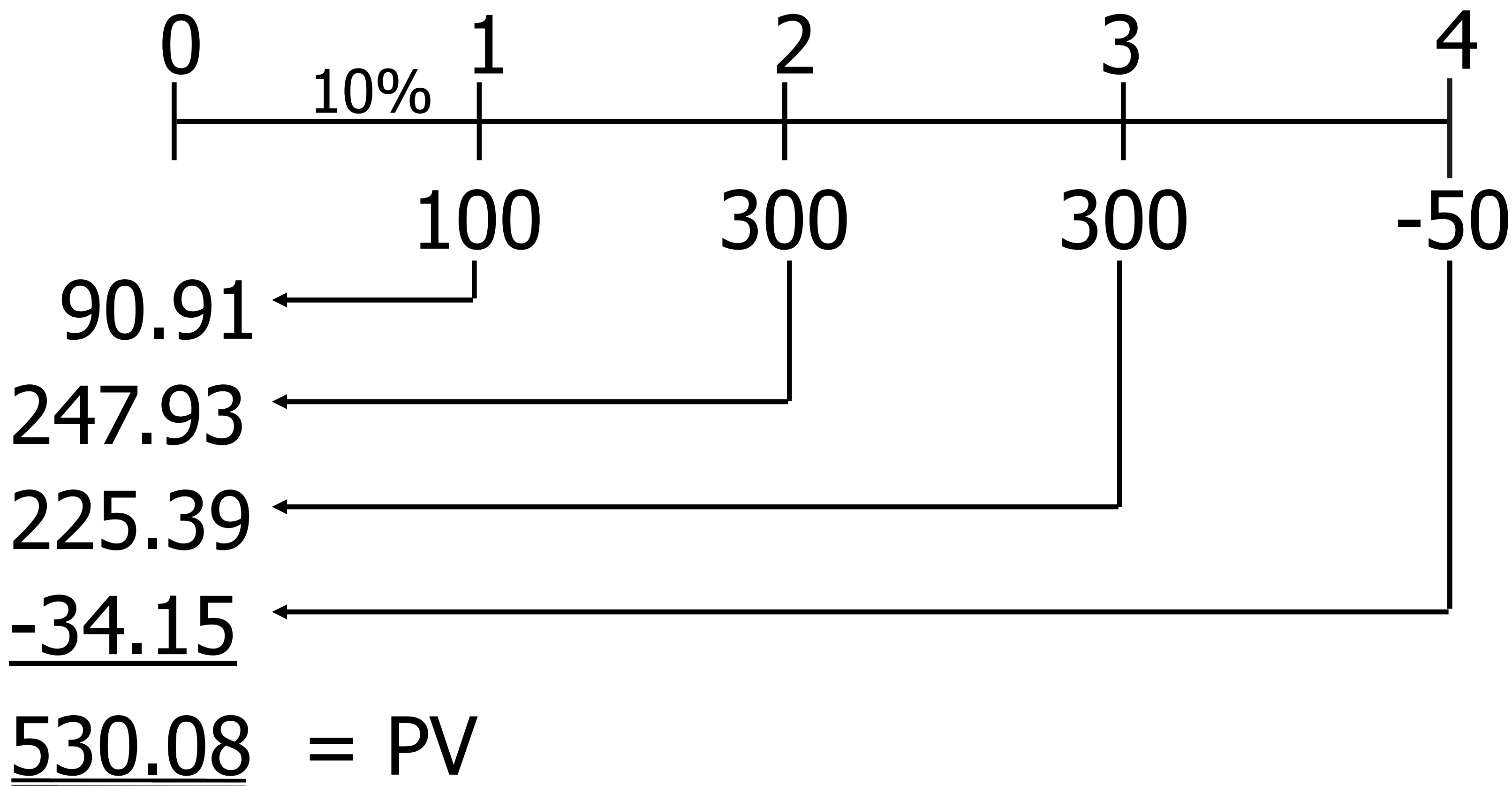
Excel Function for Annuities Due

- Change the formula to:
- =PV(0.10,3,-100,0,1)

- The fourth term, 0, tells the function there are no other cash flows. The fifth term tells the function that it is an annuity due. A similar function gives the future value of an annuity due:

- =FV(0.10,3,-100,0,1)

What is the PV of this uneven cash flow stream?



Financial calculator: HP 10bII+

- Clear all: Orange Shift key, then C All key (in orange).
- Enter number, then hit the CFj key.
- Repeat for all cash flows, in order.
- To find NPV: Enter interest rate (I/YR). Then Orange Shift key, then NPV key (in orange).

Financial calculator: HP 10bII+ (more)

- To see current cash flow in list, hit RCL CFj CFj
- To see previous CF, hit RCL CFj –
- To see subsequent CF, hit RCL CFj +
- To see CF 0-9, hit RCL CFj 1 (to see CF 1). To see CF 10-14, hit RCL CFj . (period) 1 (to see CF 11).

Financial calculator: BAII +

- Clear all cash flows: CF; 2nd CLR Work.
- CF0 displayed. Enter number, then hit the ENTER key.
- Hit the down arrow to display C01. Enter number, hit ENTER.
- F01 displayed. Usually just hit 1 ENTER. If you have several cash flows that are all the same, then use F01 to say how many you have.

Financial calculator: BAII +

- Repeat for all cash flows, in order.
- To find NPV: Hit NPV; I = will display. Enter interest rate (as a percentage, so enter 10 for 10%) ENTER; Down Arrow; Displays NPV =; hit CPT and the NPV will display.

Financial calculator: BAII + (more)

- To see current cash flow in list, hit CF
- Scroll up or down using the up and down arrows.

- Input in “CFLO” register:
  - CF0 = 0
  - CF1 = 100
  - CF2 = 300
  - CF3 = 300
  - CF4 = -50
- Enter I/YR = 10, then press NPV button to get NPV = 530.09. (Here NPV = PV.)

Excel Formula in cell A3: =NPV(10%,B2:E2)

	A	B	C	D	E
1	**0**	**1**	**2**	**3**	**4**
2		**100**	**300**	**300**	**-50**
3	**$530.09**				
4					
5					

Nominal rate (I_{NOM})

- Stated in contracts, and quoted by banks and brokers.
- Not used in calculations or shown on time lines
- Periods per year (M) must be given.
- Examples:
  - 8%; Quarterly
  - 8%, Daily interest (365 days)

Periodic rate (I_{PER})

- $I_{PER} = I_{NOM}/M$, where M is number of compounding periods per year. M = 4 for quarterly, 12 for monthly, and 360 or 365 for daily compounding.
- Used in calculations, shown on time lines.
- Examples:
  - 8% quarterly: I_{PER} = 8%/4 = 2%.
  - 8% daily (365): I_{PER} = 8%/365 = 0.021918%.

The Impact of Compounding

- Will the FV of a lump sum be larger or smaller if we compound more often, holding the stated I% constant?
- Why?

The Impact of Compounding (Answer)

- LARGER!
- If compounding is more frequent than once a year--for example, semiannually, quarterly, or daily--interest is earned on interest more often.

FV Formula with Different Compounding Periods

$$FV_N = PV\left(1 + \frac{I_{NOM}}{M}\right)^{M\,N}$$

$100 at a 12% nominal rate with semiannual compounding for 5 years

$$FV_N = PV\left(1 + \frac{I_{NOM}}{M}\right)^{M\,N}$$

$$FV_{5S} = \$100\left(1 + \frac{0.12}{2}\right)^{2x5}$$

$$= \$100(1.06)^{10} \quad = \$179.08$$

FV of $100 at a 12% nominal rate for 5 years with different compounding

FV(Ann.)	$= \$100(1.12)^{5}$	$= \$176.23$
FV(Semi.)	$= \$100(1.06)^{10}$	$= \$179.08$
FV(Quar.)	$= \$100(1.03)^{20}$	$= \$180.61$
FV(Mon.)	$= \$100(1.01)^{60}$	$= \$181.67$
FV(Daily)	$= \$100(1+(0.12/365))^{(5\times 365)}$	$= \$182.19$

Effective Annual Rate (EAR = EFF%)

- The EAR is the annual rate that causes PV to grow to the same FV as under multi-period compounding.

Effective Annual Rate Example

- Example: Invest $1 for one year at 12%, semiannual:

$$FV = PV(1 + I_{NOM}/M)^M$$

$$FV = \$1\ (1.06)^2 = \$1.1236.$$

- EFF% = 12.36%, because $1 invested for one year at 12% semiannual compounding would grow to the same value as $1 invested for one year at 12.36% annual compounding.

Comparing Rates

- An investment with monthly payments is different from one with quarterly payments. Must put on EFF% basis to compare rates of return. Use EFF% only for comparisons.
- Banks say "interest paid daily." Same as compounded daily.

EFF% for a nominal rate of 12%, compounded semiannually

$$\text{EFF\%} = \left(1 + \frac{I_{NOM}}{M}\right)^{M} - 1$$

$$= \left(1 + \frac{0.12}{2}\right)^{2} - 1$$

$$= (1.06)^2 - 1.0$$
$$= 0.1236 = 12.36\%.$$

Finding EFF with HP10BII

- Type in nominal rate, then Orange Shift key, then NOM% key (in orange).
- Type in number of periods, then Orange Shift key, then P/YR key (in orange).
- To find effective rate, hit Orange Shift key, then EFF% key (in orange).

EAR (or EFF%) for a Nominal Rate of of 12%

EAR_{Annual} = 12%.

$EAR_Q = (1 + 0.12/4)^4 - 1 = 12.55\%.$

$EAR_M = (1 + 0.12/12)^{12} - 1 = 12.68\%.$

$EAR_{D(365)} = (1 + 0.12/365)^{365} - 1 = 12.75\%.$

Can the effective rate ever be equal to the nominal rate?

- Yes, but only if annual compounding is used, i.e., if M = 1.
- If M > 1, EFF% will always be greater than the nominal rate.

When is each rate used?

I_{NOM}: Written into contracts, quoted by banks and brokers. Not used in calculations or shown on time lines.

When is each rate used? (Continued)

I_{PER}: Used in calculations, shown on time lines.

If I_{NOM} has annual compounding, then $I_{PER} = I_{NOM}/1 = I_{NOM}$.

When is each rate used? (Continued)

- EAR (or EFF%): Used to compare returns on investments with different payments per year.
- Used for calculations if and only if dealing with annuities where payments don’t match interest compounding periods.

Amortization

- Construct an amortization schedule for a $1,000, 10% annual rate loan with 3 equal payments.

Step 1: Find the required payments.

0	1	2	3
10%			
-1,000	PMT	PMT	PMT

	N	I/YR	PV	PMT	FV
INPUTS	3	10	-1000		0
OUTPUT				402.11	

Step 2: Find interest charge for Year 1.

$$INT_t = \text{Beg bal}_t\ (I)$$

$$INT_1 = \$1{,}000(0.10) = \$100$$

Step 3: Find repayment of principal in Year 1.

$$\begin{aligned} \text{Repmt} &= \text{PMT} - \text{INT} \\ &= \$402.11 - \$100 \\ &= \$302.11 \end{aligned}$$

Step 4: Find ending balance after Year 1.

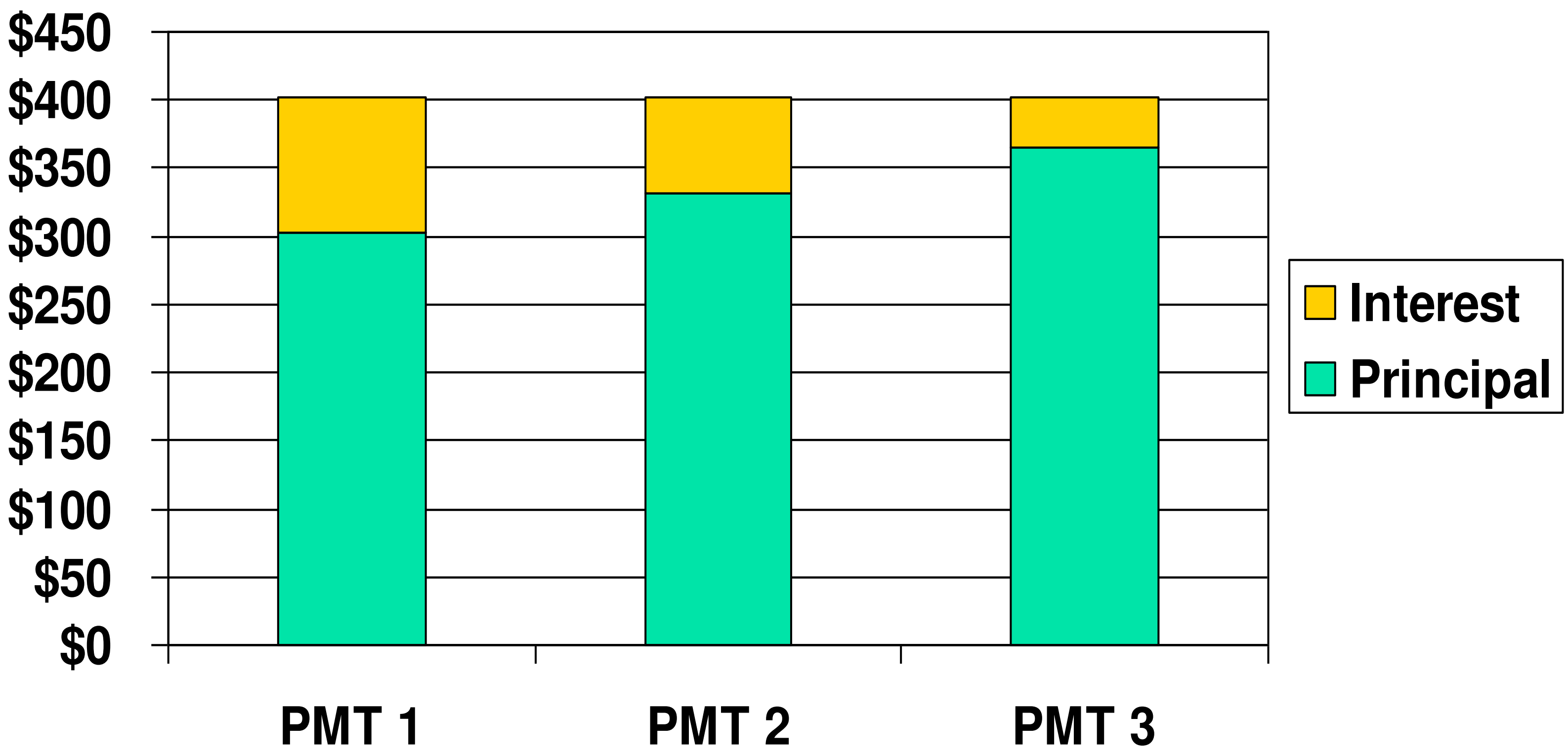
End bal = Beg bal - Repmt
= $1,000 - $302.11 = $697.89

Repeat these steps for Years 2 and 3 to complete the amortization table.

Amortization Table

YEAR	BEG BAL	PMT	INT	PRIN PMT	END BAL
1	$1,000	$402	$100	$302	$698
2	698	402	70	332	366
3	366	402	37	366	0
TOT		1,206.34	206.34	1,000	

Interest declines because outstanding balance declines.



Amortization

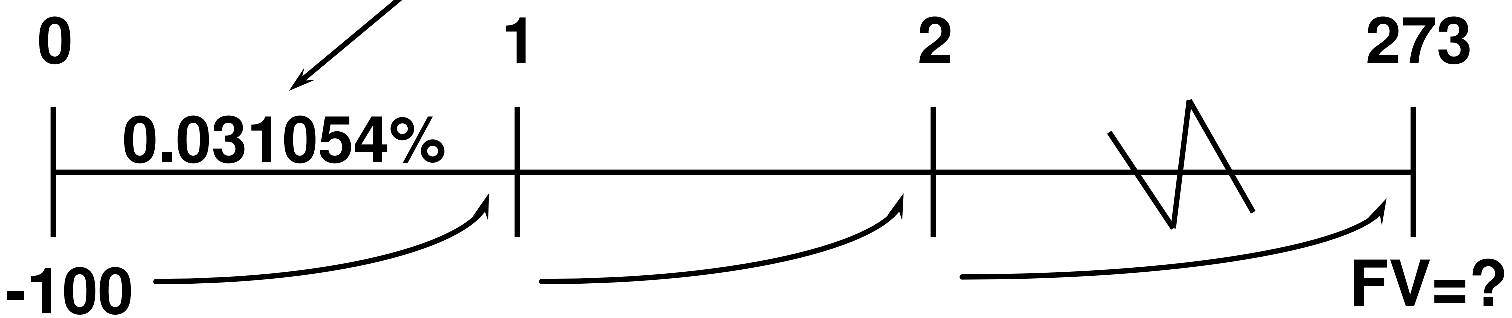
- Amortization tables are widely used--for home mortgages, auto loans, business loans, retirement plans, and more. They are very important!
- Financial calculators (and spreadsheets) are great for setting up amortization tables.

Fractional Time Periods

- On January 1 you deposit $100 in an account that pays a nominal interest rate of 11.33463%, with daily compounding (365 days).
- How much will you have on October 1, or after 9 months (273 days)? (Days given.)

Convert interest to daily rate

$$I_{PER} = 11.33463\%/365$$
$$= 0.031054\% \text{ per day}$$



Find FV

$$FV_{273} = \$100\ (1.00031054)^{273}$$

$$= \$100\ (1.08846) = \$108.85$$

Calculator Solution

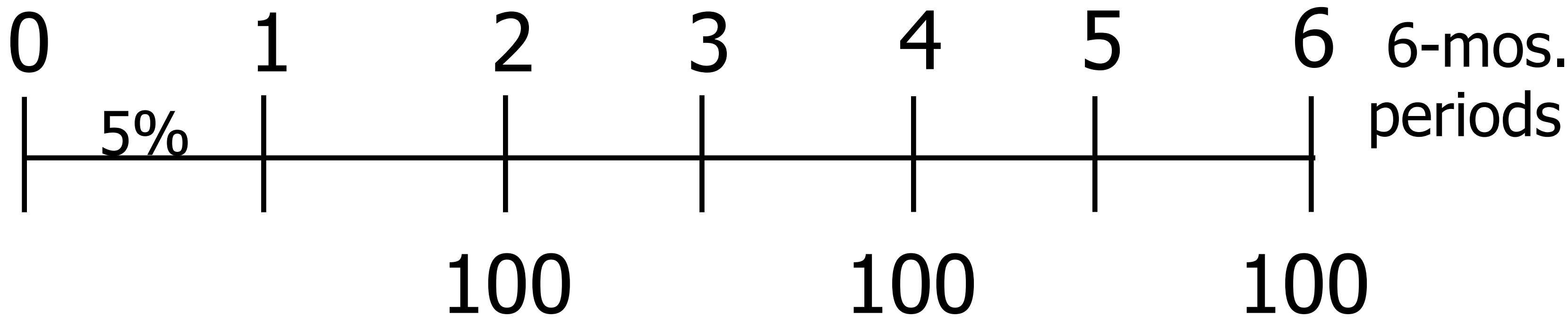
$$I_{PER} = I_{NOM}/M$$
$$= 11.33463/365$$
$$= 0.031054 \text{ per day.}$$

INPUTS	273	0.031054	-100	0	
	N	**I/YR**	**PV**	**PMT**	**FV**
OUTPUT					108.85

Non-matching rates and periods

- What's the value at the end of Year 3 of the following CF stream if the quoted interest rate is 10%, compounded semiannually?

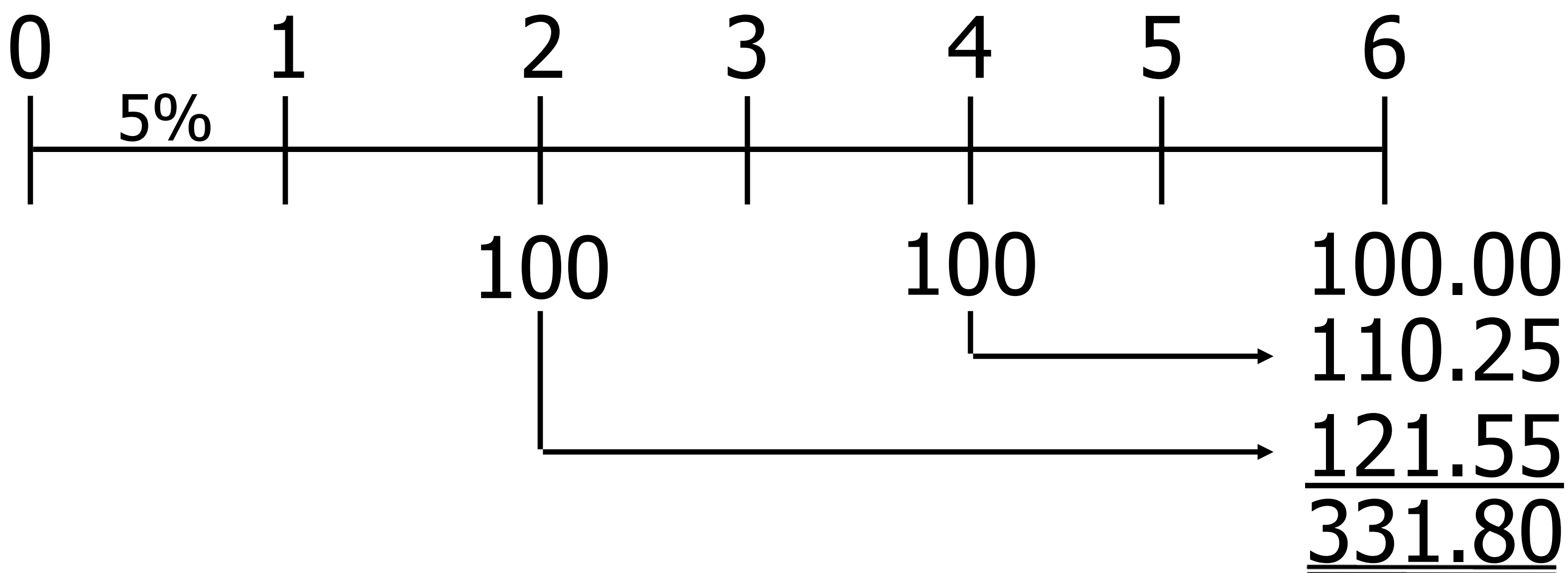
Time line for non-matching rates and periods



Non-matching rates and periods

- Payments occur annually, but compounding occurs each 6 months.
- So we can't use normal annuity valuation techniques.

1st Method: Compound Each CF



$$FVA_3 = \$100(1.05)^4 + \$100(1.05)^2 + \$100$$
$$= \$331.80$$

2nd Method: Treat as an annuity, use financial calculator

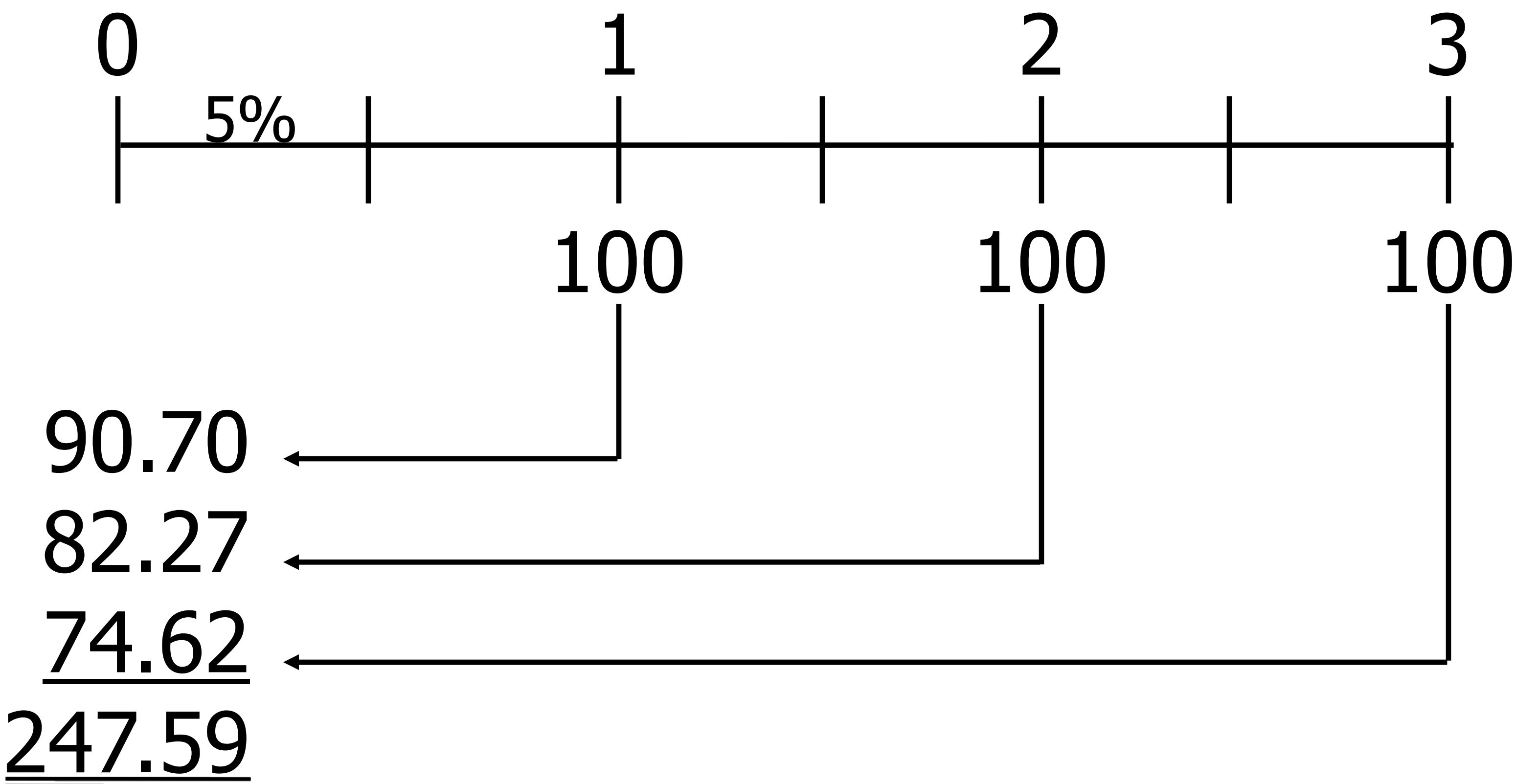
Find the EFF% (EAR) for the quoted rate:

$$\text{EFF\%} = \left(1 + \frac{0.10}{2}\right)^2 - 1 = 10.25\%$$

Use EAR = 10.25% as the annual rate in calculator.

	N	I/YR	PV	PMT	FV
INPUTS	3	10.25	0	-100	
OUTPUT					331.80

What's the PV of this stream?

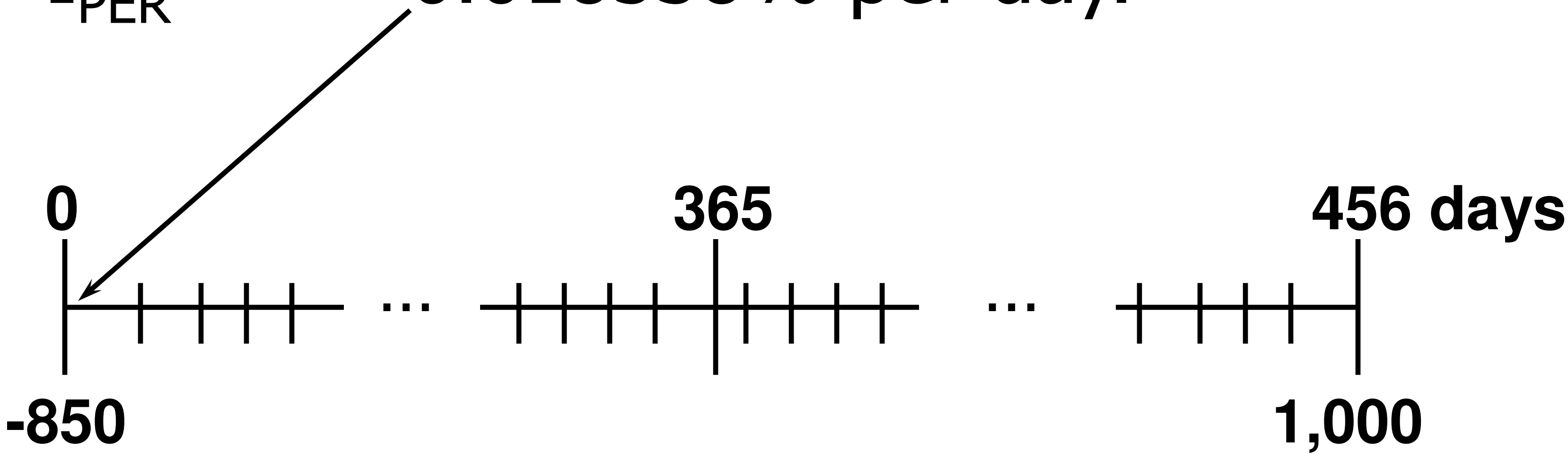


Comparing Investments

- You are offered a note that pays \$1,000 in 15 months (or 456 days) for \$850. You have \$850 in a bank that pays a 6.76649% nominal rate, with 365 daily compounding, which is a daily rate of 0.018538% and an EAR of 7.0%. You plan to leave the money in the bank if you don't buy the note. The note is riskless.
- Should you buy it?

Daily time line

I_{PER} = 0.018538% per day.



Three solution methods

- 1. Greatest future wealth: FV
- 2. Greatest wealth today: PV
- 3. Highest rate of return: EFF%

1. Greatest Future Wealth

Find FV of $850 left in bank for 15 months and compare with note's FV = $1,000.

$$FV_{Bank} = \$850(1.00018538)^{456}$$
$$= \$924.97 \text{ in bank.}$$

Buy the note: $1,000 > $924.97.

Calculator Solution to FV

$$I_{PER} = I_{NOM}/M$$
$$= 6.76649/365$$
$$= 0.018538 \text{ per day.}$$

	N	I/YR	PV	PMT	FV
INPUTS	456	0.018538	-850	0	
OUTPUT					924.97

2. Greatest Present Wealth

Find PV of note, and compare with its $850 cost:

$$PV = \$1{,}000/(1.00018538)^{456}$$
$$= \$918.95$$

Buy the note: $918.95 > $850

Financial Calculator Solution

	N	I/YR	PV	PMT	FV
INPUTS	456	6.76649/365 = .018538		0	1000
OUTPUT			-918.95		

PV of note is greater than its $850 cost, so buy the note. Raises your wealth.

3. Rate of Return

Find the EFF% on note and compare with 7.0% bank pays, which is your opportunity cost of capital:

$$FV_N = PV(1 + I)^N$$

$$\$1,000 = \$850(1 + I)^{456}$$

Now we must solve for I.

Calculator Solution

	N	I/YR	PV	PMT	FV
INPUTS	456		-850	0	1000
OUTPUT		0.035646% per day			

Convert % to decimal:

Decimal = 0.035646/100 = 0.00035646.

EAR = EFF% $= (1.00035646)^{365} - 1$

$= 13.89\%.$

Using interest conversion

$$\begin{aligned} P/YR &= 365 \\ NOM\% &= 0.035646(365) = 13.01 \\ EFF\% &= 13.89 \end{aligned}$$

Since 13.89% > 7.0% opportunity cost, buy the note.